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David B. Quanbeck

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David B. Quanbeck

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Operations Evaluation Group

CENTER FOR NAVAL ANALYSES

2000 North Beauregard Street, Alexandria, Virginia 22311

ABSTRACT

Methods for generating three dimensional aircraft trajectories necessary for quantitatively assessing aircraft tactics are documented in this report. Elements conventionally used in modeling aircraft motion are assembled to form a model governing aircraft translation, fuel use, and attitude. Assumptions on the functional dependence of the aircraft external forces and specific fuel consumption result in a system of seven equations and eleven variables governing aircraft trajectories.

To provide flexibility in prescribing aircraft trajectories, the problem of solving the equations is formulated for five separate sets of known variables. These sets include variables defining aircraft controls, velocity attitude, and velocity magnitude. Extensions to the problem formulations allow flight path normal acceleration to be prescribed, also. A method to prescribe known variables is presented that ensures continuous aircraft acceleration and angular velocity. Numerical integration, finding roots of equations, and interpolation of function values are required to solve the trajectory generation problems. Application of selected algorithms for numerical solution of the equations is discussed.

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I. INTRODUCTION

The mission effectiveness of tactical aircraft can be assessed by methods ranging from using mathematical models of the aircraft and threat weapon systems to flight testing tactics in a simulated combat environment. The former approach is useful for preliminary assessment of alternative tactics prior to employing the more costly flight testing for a more accurate assessment. Elements used to mathematically model engagements between an aircraft and a weapon system may include a model that governs aircraft motion, models of aircraft subsystems such as weapons or radar, and models of the opposing weapon systems such as surface-to-air missiles and radars. Each model reflects the inherent capabilities and limitations of each operational weapons system while the outcome of a particular engagement is an assessment of the weapon system's overall performance given the tactical employment of the systems during the engagement.

Aircraft tactics, in particular, vary widely due to the variations in the trajectory an aircrew can fly to accomplish a mission in addition to the options available for employing any of the subsystems. Quantitatively describing a particular aircraft tactic requires the ability to generate a time history of the aircraft trajectory used during the tactic. The variables describing a trajectory that are frequently needed in assessing aircraft tactics are the aircraft position, velocity, attitude, and fuel use. In general, a model for generating aircraft trajectories should provide for aircraft motion involving arbitrary three-dimensional maneuvers that can be feasibly achieved by a particular aircraft during controlled flight.

This report documents a mathematical model and solution methods that can be implemented to numerically generate aircraft trajectories required for assessing the effectiveness of tactics. The model is composed of elements of aircraft dynamics conventionally used in modeling aircraft trajectories. Specifically, six scalar equations of motion derived from the vector force equation expressing Newton's second law and an equation governing aircraft fuel flow comprise the point-mass model. These seven equations are first order differential equations governing seven variables defining the aircraft's velocity, position, and fuel use. Aerodynamic forces, engine thrust and specific fuel consumption appear in the equations. This information defines the inherent capabilities and limitations of a particular aircraft in terms of the trajectories it can feasibly achieve. Taking conventional assumptions for the functional dependence of the forces, four additional control variables defining the magnitude and orientation of the forces are needed to complete the set of variables in the model. The resulting under-determined system of seven equations and 11 variables can be solved if any four variables are prescribed over time.

Prescribing four of the variables provides control over the aircraft motion necessary to generate a particular trajectory. The choice of the variables that are prescribed also determines the procedures required to solve the system of equations. When the four control variables are prescribed, the aircraft motion is found by numerically integrating the equations of motion. If selected state variables are prescribed, unknown control variables must be found as roots of appropriate governing equations with the remaining equations integrated. In this report, the solution procedures for five different sets of prescribed variables are presented. The five sets include the set of prescribed control variables, two sets used to prescribe the attitude of an aircraft's velocity vector with selected controls, and two sets allowing the velocity magnitude and attitude to be prescribed with a selected control variable. A particular set of prescribed variables can be selected according to which set allows a particular portion of a trajectory to be most conveniently defined. A variation of the problem formulations allows the flight path normal acceleration to be prescribed instead of one of the angles defining the velocity attitude.

To numerically solve the equations, the variables prescribed as functions of time can be constructed using arbitrary functions or with a procedure presented in this report that evaluates the variables in terms of a given sequence of second time derivatives. This procedure ensures that linear continuous aircraft acceleration and angular velocity result from the prescribed variables. Algorithms for integration and finding roots of equations are required to numerically solve the equations of motion. In general, each derivative evaluation during numerical integration requires finding unknown control variables as roots of algebraic equations. Additionally, interpolation between discrete function values is required to approximate the forces and specific fuel consumption appearing in the equations.

The aircraft trajectory model presented in this report includes elements that are useful for investigating other problems in flight dynamics. Flight path parameter optimization problem formulations frequently include the equations of motion. For example, a parameter dependent on flight path variables may be minimized subject to constraint equations which include the equations of motion. Adding the moment equations governing the angular motions of the aircraft provides a model that may be used to investigate aircraft stability and control problems. In the trajectory model presented here, the moment equations are neglected and the aircraft angular motions implied during a trajectory are assumed to be feasible. The moment equations can be solved given the angular motions during a particular trajectory if this assumption is questioned.

This report consists of four sections after this introduction. The next section presents the model governing the aircraft motion. Included in this section are definitions of variables in the problem, development of the scalar equations of motion, and discussion of the assumptions on the forces and specific fuel consumption. The third section presents the individual problems formulated with the different sets of prescribed variables. The steps required to solve each problem are identified for both the general problems and the simpler cases of zero sideslip flight and symmetric flight in the vertical plane. Prescribing the velocity attitude in terms of aircraft acceleration is considered followed by a discussion of limitations and extensions of the trajectory model. Numerical methods that will be used to implement the solution of the equations are presented in the fourth section. These include a method for prescribing variables and algorithms chosen for integration, root-finding, and interpolation. Special consideration is given to application of the algorithms for solving the equations in the trajectory model. The last section of the report briefly summarizes the model and presents conclusions concerning the application of the methods for generating trajectories.

II. AIRCRAFT MATHEMATICAL MODEL

This section presents a mathematical model governing aircraft translation and fuel use. First, three reference frames are introduced and the transformations between the reference frame coordinate axes are presented. These are the inertial, the wind, and the body-fixed reference frames used for representing the forces acting on the aircraft and the motion of the aircraft. Equations for calculating angular velocities of the moving reference frames are given followed by equations for calculating the aircraft attitude. The scalar force equations of motion and a scalar equation governing fuel flow are then presented. This model, selected here to govern aircraft translation and fuel use, is the point-mass model used in trajectory analyses. The model neglects the equations governing the aircraft's angular motions about its center of gravity. This section ends with a discussion of the trajectory model. Assumptions are taken that define the dependence of the force and specific fuel consumption functions on state and control variables in the model. With these assumptions, the trajectory model consists of seven equations and 11 variables. Defining any four variables as known functions of time determines a unique trajectory. Five different sets of known variables are considered this report for prescribing aircraft trajectories. Discussion of these five sets and the corresponding problem formulations concludes this section.

This section includes material necessary to present the model governing aircraft trajectories and serves to document the equations comprising the model. More detailed development and discussion of the elements of aircraft dynamics presented here can be found in [1] and [2].

REFERENCE FRAMES

Three reference frames with right-handed coordinate systems will be used to represent aircraft forces and motion. These reference frames, the inertial frame, the wind frame, and the body fixed frame, are described below.

Inertial frame, F_I-- Newton's laws govern the motion of a body with respect to an inertial frame. In this report, the inertial coordinate system, xyz, is assumed fixed on a flat earth. Acceleration of the aircraft due to flight over the rotating curved earth is neglected in the flat earth approximation. The increase in the acceleration with aircraft speed is discussed in [1] where the flat earth approximation is considered appropriate for flight at speeds below about Mach 3. The orientation of coordinate axes is such that z is positive in the direction of positive gravity, $\tilde{g}$, which is also assumed constant. The directions of the x and y axes are arbitrary.

Wind Frame, F_W-- This is a moving frame with the origin of the axes $x_W y_W z_W$ fixed at the aircraft c.g. The x_W axis is defined to be coincident with the aircraft's velocity, $\bar{V}$, with respect to the air mass (true airspeed). The z_w axis is positive in the lower half of the aircraft plane of symmetry, downward during level flight.

Body-Fixed Frame, F_b--This moving frame also has its origin at the aircraft's c.g. and the axes $x_b y_b z_b$ are fixed with respect to the aircraft. The axis x_b is defined to be coincident with the zero-lift longitudinal axis of the aircraft, positive forward. The axis z_b is positive in the lower half of the aircraft plane of symmetry.

The position of the aircraft c.g. in the inertial frame will be noted by the vector $\bar{X}_I = [x_I, y_I, z_I]^t$. The aircraft's inertial velocity, the time derivative of $\bar{X}_I$, is then $\bar{V}_I = [\dot{x}_I, \dot{y}_I, \dot{z}_I]^t$. If air mass has a velocity $\bar{W}$ with respect to the inertial frame, then $\bar{V}_I = \bar{V} + \bar{W}$. In the following development, $\bar{W}$ will be assumed constant in time and space.

The coordinate axes of the moving reference frames are displaced by translation and rotation from the inertial axes. Components of the same vector observed from two parallel coordinate systems are equal, but angular orientations of the moving coordinate systems need to be defined to develop matrices for transforming vectors between rotated coordinate systems. The orientation of the wind and body-fixed axes are shown in figure 1 where x' y' z' is a set of axes parallel to the inertial frame.

Three Euler angles designate the orientation of the wind axes from the inertial axes. First, x' and y' are rotated about the z' axis to form an intermediate coordinate system x_1, y_1, z' where x_1

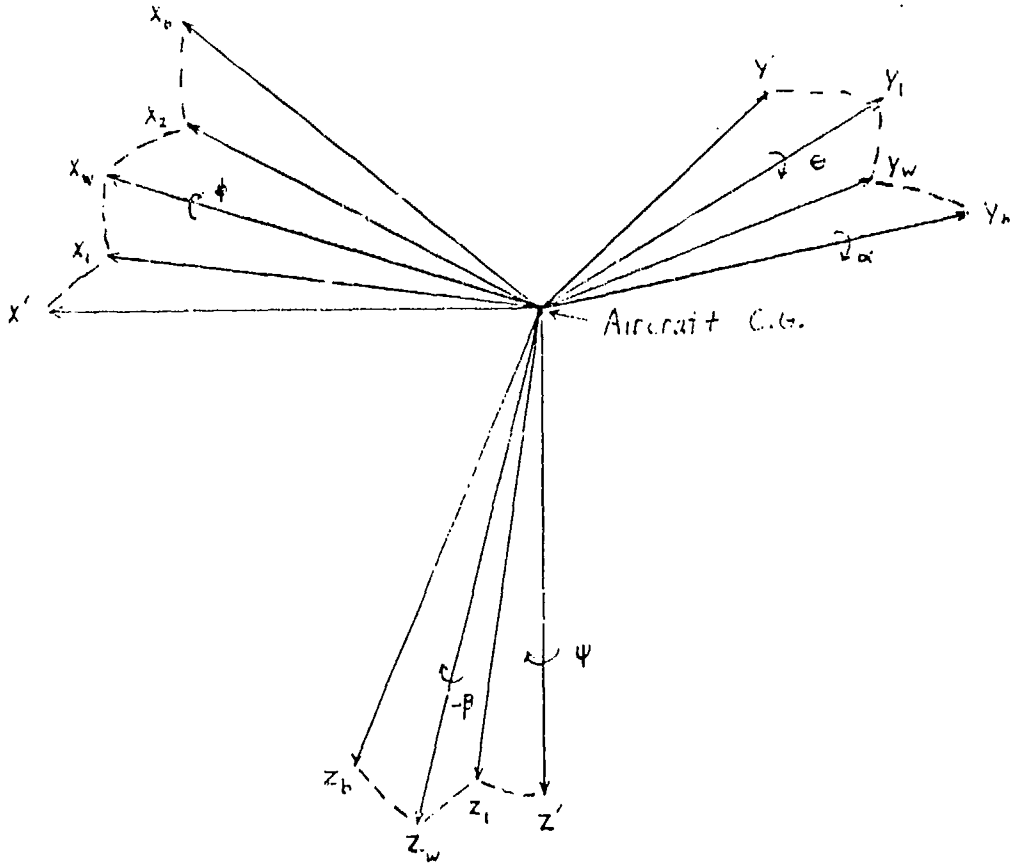

FIGURE 1: EULER ROTATIONS DEFINING WIND AND BODY AXES

is coincident with the projection of $\bar{V}$ in the horizontal plane. The angle of rotation is the velocity yaw angle, ψ. A rotation θ, the velocity pitch angle, about y_1 carries x_1 to x_w, coincident with $\bar{V}$, resulting in a second set of intermediate axes x_w, y_1, z_1. The velocity roll angle, ϕ, is the final rotation about x_w to carry z_1 into the aircraft plane of symmetry thus forming the wind axes $x_w y_w z_w$. Each of the Euler rotations is a rotation of two axes in a plane, so three vector coordinate transformations about a single axis occur in sequence. These rotations result in a matrix, L_{wI}, to transform a vector $\bar{A}_I$ expressed on the inertial axes to the same vector $\bar{A}_w = L_{wI}\bar{A}_I$ expressed in F_w.

$$L_{wI} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \qquad (1)$$

L_{wI} is an orthogonal matrix, its inverse is equal to its transpose. Thus, the transformation from F_w to F_I is given by $A_I = L_{Iw}A_w$, where $L_{Iw} = L_{wI}^T$.

The orientation of the body-fixed axes with respect to the wind axes is also defined by two Euler angles. A rotation, $-\beta$, about z_w results in the $x_2 y_b z_w$ intermediate axes. The quantity β is the sideslip angle and the $x_2 z_w$ plane is also the aircraft plane of symmetry. Rotating the coordinates in this plane about y_b through the angle of attack α yields the aircraft body-fixed axes $x_b y_b z_b$. The orthogonal transformation matrix resulting from these two rotations is:

$$L_{bw} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \tag{2}$$

Expressions for the angular velocities of the moving coordinate systems are developed next. Of particular interest is the angular velocity of the axes $x_w y_1 z_1$ to be used to express the aircraft inertial acceleration in this coordinate system. The aircraft angular velocity, with components being the aircraft yaw, pitch, and roll rates, is the angular velocity of $x_b y_b z_b$. Monitoring the value of these components will be useful, especially during maneuvers involving rapid changes of aircraft attitude.

The total angular velocity vector of a moving coordinate system with respect to the inertial frame is the sum of the angular velocity vectors due to the time rate of change of each of the Euler rotations. For the axes $x_w y_1 z_1$, the angular velocity, $\bar{\omega}_{w'}$, due to the rates $\dot{\theta}$ and $\dot{\psi}$ is

$$\bar{\omega}_{w'} = \dot{\theta}\bar{j}_1 + \dot{\psi}\bar{k}' \tag{3}$$

Here, $\bar{j}_1$ and $\bar{k}'$ are the unit vectors on the y_1 and z' axes, respectively. Observing that $\bar{k}' = -\sin\theta\,\bar{i}_w + \cos\theta\,\bar{k}_1$, where $\bar{i}_w$ and $\bar{k}_1$ are unit vectors on x_w and y_1, gives $\bar{\omega}_{w'}$, expressed on $x_w y_1 z_1$ as

$$\bar{\omega}_{w'} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} \tag{4}$$

Calculating the angular velocity of the body axes requires adding the components due to $\dot{\phi}$, $-\dot{\beta}$, and $\dot{\alpha}$ to the two components summed in equation 3. First, the angular velocity of the wind axes $x_w y_w z_w$ can be found with the components expressed in terms of the wind axes coordinates. Employing the approach used to obtain equation 4, the resulting wind axes angular velocity is

$$\bar{\omega}_w = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \\ -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta \end{bmatrix} \tag{5}$$

The angular velocity vectors due to $-\dot{\beta}$ and $\dot{\alpha}$ are next expressed in terms of the body-fixed coordinate system. Summing this vector with $\bar{\omega}_w$ transformed to the body-fixed frame gives the following expression for calculating the aircraft angular velocity.

$$\bar{\omega}_b = \begin{bmatrix} p_b \\ q_b \\ r_b \end{bmatrix} = \begin{bmatrix} \dot{\beta}\sin\alpha \\ \dot{\alpha} \\ -\dot{\beta}\cos\alpha \end{bmatrix} + L_{bw}\bar{\omega}_w \tag{6}$$

The components p_b, q_b, and r_b are the aircraft yaw, pitch and roll rates, respectively.

Knowledge of an aircraft's attitude with respect to the inertial axes is frequently important in assessing a trajectory. An aircrew's field-of-view, aircraft sensor and weapons employment envelopes, and the aircraft's aspect as seen by a second observer are examples of quantities dependent on the aircraft attitude. The orientation of the body-fixed axes from the inertial axes can be defined by three Euler angles ψ_b, θ_b, and ϕ_b which are the aircraft yaw, pitch, and roll angles. These angles are analogous to the wind axes Euler angles and, therefore, a transformation matrix, L_{bI}, results identical to equation 1 except that ψ_b, θ_b, and ϕ_b replace the corresponding wind axes Euler angles. To calculate the aircraft attitude, terms of L_{bI} can be equated to terms in the matrix, $\{\ell_{ij}\}$, equal to the product $L_{bw} \cdot L_{wI}$. This results in the expressions below for the body-fixed Euler angles.

$$\psi_b = \tan^{-1} \frac{\ell_{12}}{\ell_{11}} \tag{7a}$$

$$\theta_b = -\sin^{-1} \ell_{13}, \quad -\pi/2 < \theta_b < \pi/2 \tag{7b}$$

$$\phi_b = \tan^{-1} \frac{\ell_{23}}{\ell_{33}} \tag{7c}$$

Above, the terms of $\{\ell_{ij}\}$ are the results of evaluating trigonometric functions of α, β, and the wind axes Euler angles. The signs of the arguments in the inverse tangent functions above will determine the appropriate quadrants of ψ_b and ϕ_b.

EQUATIONS OF MOTION

This section presents the scalar equations of motion governing aircraft translation. First, three scalar force equations governing aircraft velocity are formulated on the moving axes, $x_w y_1 z_1$. The aircraft position in the inertial frame is governed by three additional equations. A single equation governing aircraft fuel flow completes the model.

A derivation of the force equation governing aircraft motion is presented in [2]. The resulting vector equation consistent with the flat earth approximation is

$$\bar{T} + \bar{A} + m\bar{g} = m\bar{a}_I . \tag{8}$$

where
$\bar{T}$ = aircraft thrust
$\bar{A}$ = aerodynamic force
$\bar{g}$ = acceleration of gravity
m = aircraft mass
$\bar{a}_I$ = inertial acceleration of the aircraft mass center.

Three scalar equations for the time derivatives $\dot{V}$, $\dot{\psi}$, and $\dot{\theta}$ will be found next by expressing the components of the equation 8 on the moving coordinate system $x_w y_1 z_1$. In doing so, each of three derivative terms will appear in only one equation, a convenient form for numerical integration. However, the components of $\bar{T}$ and $\bar{A}$ will be summed first on the wind axes and then transformed to $x_w y_1 z_1$.

Net thrust, T, is assumed to act in the aircraft plane of symmetry, $x_b z_b$, at a fixed angle ε elevated from the aircraft longitudinal axis, x_b. Transforming the thrust vector from the body fixed axes to the wind axes gives

$$\bar{T} = L_{wb} \begin{bmatrix} T\cos\varepsilon \\ 0 \\ -T\sin\varepsilon \end{bmatrix} = \begin{bmatrix} T\cos\beta\cos(\alpha+\varepsilon) \\ -T\sin\beta\cos(\alpha+\varepsilon) \\ -T\sin(\alpha+\varepsilon) \end{bmatrix} . \tag{9}$$

The three components of $\bar{T}$ above will be noted as T_{x_w}, T_{y_w}, and T_{z_w}. The aerodynamic force vector components are defined in the wind axes as drag, side force and lift on the x_w, y_w, and z_w axes, respectively. All three components are assumed to act in negative direction of their respective axes giving

$$\bar{A} = -\begin{bmatrix} D \\ C \\ L \end{bmatrix} . \tag{10}$$

The wind axes are obtained from $x_w y_1 z_1$ by a single rotation ϕ about the x_w axis. Then $\bar{T}+\bar{A}$ is transformed to $x_w y_1 z_1$ by

$$\bar{T} + \bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T_{x_w} - D \\ T_{y_w} - C \\ T_{z_w} - L \end{bmatrix} = \begin{bmatrix} T_{x_w} - D \\ \cos\phi(T_{y_w} - C) - \sin\phi(T_{z_w} - L) \\ \sin\phi(T_{y_w} - C) + \cos\phi(T_{z_w} - L) \end{bmatrix} \tag{11}$$

The aircraft's weight, $m\bar{g}$, has only one non-zero component, expressed on the coordinates $x'y'z'$, equal to $mg\bar{k}'$. As in the development of equation 4, $\bar{k}'$ can be replaced by $-\sin\theta\,\bar{i}_w+\cos\theta\,\bar{k}_1$. Therefore, aircraft's weight expressed on $x_wy_1z_1$ is given by

$$m\bar{g} = mg \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix} . \tag{12}$$

Finding the aircraft's inertial acceleration remains to complete the scalar force equations. Recalling $\bar{V}_I = \bar{V} + \bar{W}$ and that $\bar{W}$ is assumed to be constant, the acceleration is then the time derivative of $\bar{V}$, $d\bar{V}/dt$, observed from the inertial frame. However, the acceleration vector is to be expressed on the moving frame $_wy_1z_1$. In terms of $\bar{V}$ and $\bar{\omega}_w$, as observed from $x_wy_1z_1$, the inertial acceleration expressed on $x_wy_1z_1$ is the sum of the derivative of $\bar{V}$ and $\bar{\omega}_w x\bar{V}$. By definition, $\bar{V}$ has one component on $x_wy_1z_1$, $V\bar{i}_w$. Using equation 4 defining the components of $\bar{\omega}_w$, gives the inertial acceleration vector on $x_wy_1z_1$ as

$$\bar{a}_I = \begin{bmatrix} \dot{V} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} x \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{V} \\ V\dot{\psi}\cos\theta \\ -V\dot{\theta} \end{bmatrix} . \tag{13}$$

Having developed the components of the vectors in equation 5 expressed on $x_wy_1z_1$, three equivalent scalar equations can be written. Solving for the derivative terms in the acceleration components gives

$$\dot{V} = \frac{1}{m}[T_{x_w} - D] - g\sin\theta \qquad (14a)$$

$$\dot{\psi} = \frac{1}{mV\cos\theta}[\cos\phi(T_{y_w} - C) - \sin\phi(T_{z_w} - L)] \qquad (14b)$$

$$\dot{\theta} = -\frac{1}{mV}[\sin\phi(T_{y_w} - C) + \cos\phi(T_{z_w} - L)] - \frac{g}{V}\cos\theta \qquad (14c)$$

These three first order ordinary differential equations govern the magnitude and orientation of $\bar{V}$ at each time instant. The equations governing the aircraft's position in the inertial frame, $\bar{X}_I = [x_I, y_I, z_I]^t$, are obtained from $d\bar{X}_I/dt = \bar{V}_I$. In the inertial coordinate system, $\bar{V}_I$ is the sum of $L_{Iw}\bar{V}$ and the velocity of the air mass, $\bar{W} = [w_x, w_y, w_z]^t$.

$$\dot{x}_I = V\cos\theta\cos\psi + w_x \qquad (15a)$$

$$\dot{y}_I = V\cos\theta\sin\psi + w_y \qquad (15b)$$

$$\dot{z}_I = -V\sin\theta + w_z \qquad (15c)$$

To relate fuel use to aircraft motion, one additional equation will complete the model of aircraft flight expressed by equations 14a through 15c. As fuel is burned to produce thrust, the mass of the aircraft decreases. A discussion of turbojet and turbofan engines is presented in [2] giving the following general representation for the mass flow thrust relationship.

$$\dot{m} = -cT/g \qquad (16)$$

$$c = \text{specific fuel consumption}$$

DISCUSSION OF THE EQUATIONS OF MOTION

Equations 14a through 16 constitute the model of aircraft motion assumed for the remainder of this report. Seven first order ordinary differential equations govern the seven state variables (V, ψ, θ, x_I, y_I, z_I, m). The remaining variables include the angles (α, β, ϕ), the thrust and aerodynamic forces, and the specific fuel consumption. In the discussion below, assumptions are made about the functions defining the forces and the specific fuel consumption. These assumptions reduce the number of variables in the model. Then, five alternative problem formulations are presented. The alternatives arise from prescribing different sets of known variables over time and solving for the remaining variables.

Aerodynamic forces are usually represented by dimensionless coefficients obtained by dividing the forces by the product of dynamic pressure and a reference area. The coefficients are functions of aircraft attitude, shape, Mach number, and Reynold's number. Variations in the forces due to Reynold's number effects usually can be neglected, see [1]. Furthermore, the drag and lift will be assumed independent of the sideslip angle and the side force will be assumed independent of the angle of attack. With these assumptions the drag, side force, and lift coefficient functions are

$$C_D(M, \alpha) = 2D/(\rho V^2 S) \tag{17a}$$

$$C_C(M, \beta) = 2C/(\rho V^2 S) \tag{17b}$$

$$C_L(M, \alpha) = 2L/(\rho V^2 S) \tag{17c}$$

where $M = V/a$ = Mach number

a = speed of sound

ρ = atmospheric density

S = reference area (wing area)

The three force coefficient functions will be numerically approximated by interpolating between values of the functions stored in two dimensional data sets. This data will apply to a given aircraft shape. Variation in aircraft shape, by carrying external stores or changing wing sweep angle, may significantly change a force coefficient function. Either corrections to the function values or increasing the dimension of the domain of the function are required. Extensions of the functions can be determined for particular aircraft and will not be considered further here.

Thrust and specific fuel consumption of turbofan and turbojet engines can also be represented by dimensionless coefficients presented in [2]. The coefficients are assumed independent of Reynold's number and the angles α and β. Engine performance is then defined by the thrust and specific fuel consumption coefficients below.

$$K_T(M,\ n_c) = \frac{T}{pS_e} \tag{18a}$$

$$K_C(M,\ n_c) = \frac{ca_*^2}{ag} \tag{18b}$$

where p = atmospheric pressure

S_e = reference area

a_* = speed of sound at the tropopause

n_c = corrected engine speed = $na_*/(an_{max})$

n = engine rotor rpm

Like the aerodynamic force coefficients, the above functions can be approximated by two dimensional data sets for a given aircraft. Also, the functions can be used in an arbitrary atmosphere since the speed of sound is proportional to the square root of the temperature of the air. However, in [3], a discussion of engine modeling states that approximating T and c to be strictly proportional to p and a will result in errors. These errors result from Reynold's number effects and deviations from the assumed thermodynamic properties of the engine airflow. If the desired accuracy requires these effects to be included, then pressure altitude, with a standard day atmosphere assumed, can be added to the domain of the functions. If so, equations 18a and 18b can still be used to approximate engine performance in atmospheric conditions deviating from a standard day at a given pressure altitude. In this report, equations 18a and 18b will be assumed to model aircraft engine thrust and specific fuel consumption.

The aircraft model defined by equations 14a through 16 can now be interpreted in view of the assumed functional dependence of the forces and specific fuel consumption. Assuming the properties of the atmosphere are known functions of altitude, $-z_I$, the model consists of seven equations determined by eleven variables. The variables can be grouped into the seven state variables governed by the seven differential equations, $(V, \psi, \theta, x_I, y_I, z_I, m)$, and four control variables $(\alpha, \beta, \phi, n_c)$. In an aircraft dynamics model including the moment equation, as in [1], α, β, and ϕ appear as state variables primarily controlled by deflections of the elevator, rudder and ailerons, respectively. The corrected engine speed is controlled by the throttle position.

With seven equations and eleven variables, the problem of generating an aircraft trajectory can be solved numerically by prescribing any four variables as known functions over time and defining initial conditions for the state variables. Of all the possible combinations of four known variables, five different combinations will be considered in this report. The first set will be defined by prescribing the four control variables $(\alpha, \beta, \phi, n_c)$. Then the seven equations are integrated to find aircraft velocity, position and fuel use. Two more combinations of known variables considered are defined by prescribing the attitude of the velocity vector, the engine control, and either ϕ or β. Numerical solutions given either of these sets, $(\psi, \theta, \phi, n_c)$ or $(\psi, \theta, \beta, n_c)$, requires solving equations 14b and 14c for the roots (α, β) or (α, ϕ) and integrating the remaining equations. The last two sets of known variables are defined by prescribing the velocity vector and either ϕ or β, that is (V, ψ, θ, ϕ) or (V, ψ, θ, β). Then the three force equations, 14a, 14b, and 14c, are solved for the roots (α, β, n_c) or (α, ϕ, n_c). Integrating equations 15a through 16 gives aircraft position and fuel use.

In summary, any one of the sets of variables can be prescribed to formulate the problem of solving equations 14a through 16. Although any one set is sufficient to solve for any feasible aircraft trajectory, these alternative problem formulations allow flexibility in prescribing aircraft motion. With this approach, the several equations can be solved for a series of aircraft maneuvers comprising a complete trajectory. Each maneuver can be generated by prescribing the set of variables with which the maneuver is most conveniently defined. Example flight conditions easily defined with the different sets of known variables are given in the next section.

III. TRAJECTORY GENERATION PROBLEM FORMULATIONS

Each set of prescribed variables discussed at the end of the last section determines a different formulation for the problem of numerically solving equations 14a through 16. The five different problems are discussed in this section in terms of the procedures necessary to solve the problems. When state variables are known variables, control variables must be found in general by solving simultaneous equations. For zero sideslip flight and symmetric flight in the vertical plane, particular cases of interest, solving the sets of algebraic equations simplifies. In addition to discussing the different problem formulations, examples of specific aircraft maneuvers conveniently formulated in each case are given. A trajectory can be constructed by a sequence of maneuvers each prescribed with one set of known variables. Later in this section, expressions for prescribing velocity attitude angles in terms of the acceleration normal to the flight path are presented. Finally, limitations of the aircraft trajectory model are considered followed by a brief discussion of aircraft dynamics problems that may be solved by alternative formulations or extensions of the model.

PRESCRIBED CONTROLS

In this case, the control variables $(\alpha, \beta, \phi, n_c)$ are defined as known functions of time. Given initial values for the state variables, equations 14a through 16 are integrated over time giving aircraft velocity, position and fuel use. To evaluate the derivatives of the state variables for numerical integration the variables and functions appearing in equations 14a-c and 16 must be evaluated.

Several specific steps are required prior to calculating the state variable derivatives at any time, t. First the known variables, $\alpha(t)$, $\beta(t)$, $\phi(t)$, and $n_c(t)$ have to be evaluated. For example, they may be numerically evaluated from analytic functions provided for a particular problem or evaluated by interpolation between discrete points. Then the properties of air at the current altitude and the Mach number must be calculated. The aerodynamic force, thrust, and specific fuel consumption coefficients are evaluated by interpolation between stored discrete function values. These steps will be required in all five problem formulations. Finally, equations 14a through 16 for the state variable derivatives are evaluated. The numerical algorithms to accomplish each step are presented later in a separate section.

When $\beta = 0$ the side force is also zero and this condition together with $\sin\phi = 0$ results in symmetric flight in the vertical plane. Equations 14b (and 15b if $\psi = 0$, $w_y = 0$) are removed from the system of equations reducing the integration problem. These assumptions will be convenient in calculating fuel use over two-dimensional trajectories with (α, n_c) prescribed, for example, simply as constants. For flight in three dimensions, control schedules may be formulated, for example, by modifying control schedules found from solving the equations with a different set of known variables. A general extension of the problem formulation would be to evaluate controls during the integration based on the state of the aircraft motion. For example, fuel use during constant lift coefficient trajectories can be evaluated by appropriate selection of α as the integration proceeds.

PRESCRIBED VELOCITY ATTITUDE AND ENGINE CONTROL

Selecting either of the sets $(\psi, \theta, n_c, \beta)$ or $(\psi, \theta, n_c, \phi)$ as the known variables will, in general, require solving equations 14b and 14c for the unknown controls α and ϕ or β. Initial values of the remaining variables (V, x_I, z_I, y_I, m) must also be defined. At any time t the first step in the solution is to evaluate $\psi(t)$, $\theta(t)$, $n_c(t)$, and $\beta(t)$ or $\phi(t)$. Values for the derivatives of the known state variables $\dot{\psi}(t)$ and $\dot{\theta}(t)$ must also be evaluated. Then, the unknown controls, (α, ϕ) or (α, β) that satisfy equations 14b and 14c have to be found. The Newton-Raphson root-finding algorithm selected for this step in the solution is presented later. In general, the equations will have to be solved simultaneously, and repeated evaluations of the forces in the equations are required during the iterative root-finding algorithm. Once the roots are found, the derivative of the state variables in equations 14a and 15a through 16 are evaluated for numerical integration.

Two special cases of interest are symmetric flight in the vertical plane and zero sideslip flight. Symmetric flight in the vertical plane results when the set of known variables includes $\dot{\psi} = 0$ and $\sin\phi = 0$ or $\beta = 0$. Then, the problem reduces to finding α as the root of 14c and integrating 14a and 15a through 16. Defining ψ and w_y equal to zero further simplifies the problem deleting 15b from the integrated equations. Zero sideslip flight results from defining $\beta = 0$ and the solution simplifies as ϕ can be solved independently of α. Solving equation 14b and 14c for ϕ gives

$$\phi = \tan^{-1}[\dot{\psi}\cos\theta/(\dot{\theta} + \frac{g}{V}\cos\theta)] \tag{19}$$

The appropriate quadrant of ϕ is determined by the signs of the numerator and denominator of the arctangent argument. Equation 14c is then solved for the root α and integration of 14a and 15a through 16 proceeds as in the general case.

Selecting either $(\psi, \theta, n_c, \beta)$ or $(\psi, \theta, n_c, \phi)$ as known variables will be useful for finding aircraft velocity, position, and fuel use during flight easily described by the direction of the aircraft velocity vector. Symmetric flight in the vertical plane can be assumed when generating aircraft maneuvers such as level acceleration, climbs, and pull-ups from level flight. Zero sideslip flight may be assumed in modeling turning flight as in level, climbing, or descending turns. Prescribing non-zero β can generate motion such as turns with sideslip. Prescribing ϕ would be useful, for example, when rolling the aircraft to the inverted attitude prior to the transition from a climb into a dive.

PRESCRIBED VELOCITY VECTOR

Prescribing the magnitude and direction of the aircraft's velocity vector over time is possible when (V, θ, ψ, β) or (V, θ, ψ, ϕ) are selected as known variables. Given initial values for the remaining state variables, the solution requires solving 14a, 14b, and 14c for the roots α, n_c and ϕ or β and then integrating equations 15a through 16. The solution steps in this case differ from those of the previous problem formulation in that three simultaneous equations, instead of two equations, must be solved followed by integration of 15a through 16. The conditions determining symmetric flight in the vertical plane and zero sideslip flight in the preceding case also apply in these problem formulations. For both symmetric and zero sideslip flight, the root-finding problem reduces to solving 14a and 14c simultaneously for α and n_c. For zero sideslip flight, equation 19 gives the value of ϕ needed to solve 14a and 14c.

Prescribing the velocity vector is particularly useful in solving for the controls and fuel use associated with steady flight conditions $(\dot{V} = 0)$ which are conveniently expressed by (V, θ, ψ, β) or (V, θ, ψ, ϕ). Constant velocity turns, climbs and level flight are typical examples. Specifying the velocity vector also provides a convenient way to start a trajectory from a steady flight condition prior to generating aircraft maneuvers.

The five different sets of known variables considered have been discussed in terms of three general problem formulations. These three problems are characterized by particular equations that must be solved for unknown control variables and those to be integrated to find the state variables. Methods for prescribing the known variables as functions of time have not been discussed. A method useful for prescribing any of the known variables is presented in the next section. An extension to the general problem formulations resulting when θ and ψ are prescribed variables will be presented next that will allow additional flexibility in prescribing aircraft maneuvers. The extended formulations allow θ or ψ to be replaced by the flight path normal acceleration in the sets of prescribed variables.

FLIGHT PATH NORMAL ACCELERATION

Aircraft motion involving changes in velocity yaw or pitch angles requires forces acting on the aircraft often significantly larger than the forces encountered during steady flight. These forces should not exceed aircraft structural limits and the resulting acceleration should not exceed acceleration tolerable by aircrews. The magnitude of the acceleration normal to the aircraft flight path (in the y_1z_1 plane) can be found from the acceleration components on y_1 and z_1 as

$$a = V(\dot{\theta}^2 + \dot{\psi}^2\cos\theta^2)^{1/2} \tag{20}$$

Prescribing $a(t)$ in place of either $\psi(t)$ or $\theta(t)$ is a convenient way to characterize certain maneuvers, especially those to be limited by acceleration magnitude. Depending on whether $\theta(t)$ or $\psi(t)$ is prescribed with $a(t)$ the other can be found from the appropriate equation below

$$\dot{\theta} = \pm\left(\frac{a^2}{V^2} - \dot{\psi}^2\cos^2\theta\right)^{1/2} \tag{21}$$

$$\dot{\psi} = \pm \frac{1}{\cos\theta}\left(\frac{a^2}{V^2} - \dot{\theta}^2\right)^{1/2} \tag{22}$$

The appropriate sign for the derivatives $\dot{\theta}(t)$ and $\dot{\psi}(t)$ must be prescribed with $a(t)$ since either sign may produce feasible aircraft motion. The initial value of the state variable to be found must be defined, then numerical integration of equations 21 or 22 gives the value of the state variable over time. Either $\psi(t)$ or $\theta(t)$ can be replaced by $a(t)$ in any of the four known sets of variables in which they appear. Essentially, the new variable $a(t)$ is added to the original 11 variables in the problem and an additional equation, either 21 or 22, is added to the set of equations to be integrated. To solve equations 14b and 14c for unknown control variables at any time, $\dot{\theta}(t)$ or $\dot{\psi}(t)$ are evaluated with the above equations. Therefore, the solution steps outlined to solve each of the four problems with known ψ and θ still apply when $a(t)$ is prescribed.

LIMITATIONS OF THE AIRCRAFT MODEL

The model assumed to govern aircraft motion has inherent limitations that must be considered in its application. Specifically, values of the prescribed variables and the resulting aircraft motion must be considered in view of constraints on the aircraft motion not implicit in the model The moments an aircraft can physically achieve at any flight condition are an important set of constraints. Rapid changes in the aircraft attitude require large moments, so these constraints may be violated during high angular rate aircraft maneuvers. Calculating aircraft angular velocity during a trajectory, as given in equation 6, allows the angular rates to be monitored. Aircraft motion at large angles of attack or sideslip generally cannot be predicted with the trajectory model since problems in maintaining controlled flight can develop in these flight regimes. Additional constraints, due to structural and engine operating limits can often be represented by a feasible flight envelope constructed as a function of Mach number and pressure altitude. These are usually found in individual aircraft operations manuals. A more detailed discussion of constraints frequently encountered in trajectory analyses is presented in [2].

The accuracy of calculated fuel use will be dependent on the accuracy of the force and specific fuel consumption data as well as the accuracy of the approximations required for numerical solution of the equations governing flight. Flight test results or fuel flow data from operations manual provide data to validate the fuel use calculations. In certain applications of this model, for example, when comparing the fuel use of different trajectories, the relative difference in fuel use values is of primary importance. If absolute fuel flow values are to be used, as in mission planning, they should be used conservatively.

EXTENSIONS OF THE AIRCRAFT MODEL

Equations 14c through 16 have been formulated to be solved as five different trajectory generation problems. These equations provide the basis for solving several other problems of interest in aircraft dynamics. Flight path parameters or functions defined in terms of state and control variables can be optimized using elements of the model described here. Typical problems include minimizing fuel flow with respect to time or distance and maximizing climb rate subject to a set of constraint equations. The constraint equations may be for example, the force equations for symmetric flight in the vertical plane. A number of optimization problems are formulated in [2]. Example problems are also formulated in [4] and numerical methods for solving such problems are presented.

Problems in the area of aircraft stability and control are formulated in part with the force equations of motion. The moment equations governing the angular motions of the aircraft are added to these equations. Depending on assumptions about the forces and moments, the force and moment equations may be coupled and require simultaneous solution. In the trajectory model presented here, the equations are assumed independent. The moments acting on the aircraft can be evaluated after a trajectory solution when validating the feasibility of a high rate maneuver is desired. A development of the moment equations and examples of their application are presented in [1].

IV. NUMERICAL METHODS

The numerical algorithms required to solve equations 14a through 16 have been identified in the previous section. First, prescribed variables have to be evaluated to solve the equations. In general, these may be arbitrary functions constructed for a particular problem. However, one method is presented in this section for simply prescribing variables as functions of time. Algorithms for solving algebraic equations, integrating differential equations, and interpolation are also needed in the solution. The algorithms selected for this problem are described with discussion of their application in the solution of equations 14a through 16. Finally, an expression for approximating derivatives of control variables and the model of the atmosphere to be used are given.

PRESCRIBING KNOWN VARIABLES

To solve the equations of motion, four variables have to be defined as functions of time. The variables that can be prescribed include α, β, n_c, ϕ, V, ψ, θ and a. Not only the values of the prescribed variables are needed, but the values of the time derivatives must be calculated for all the variables except a and n_c. The derivatives $\dot{V}$, $\dot{\theta}$, and $\dot{\psi}$ appear in the acceleration components of equations 14a through 14c while $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\phi}$ are needed to calculate aircraft angular velocity using equation 6.

Furthermore, the time derivatives of the prescribed variables should be continuous functions of time to ensure that acceleration and angular velocity of the aircraft are continuous. This requirement arises from neglecting the aircraft moment equations governing the angular motions. Variables with linear continuous first derivatives can be constructed by defining a sequence of constant second derivatives over time. Let $u(t)$ denote a variable to be prescribed with initial values u_0 and $\dot{u}_0$ given at time t_0. Assume $\ddot{u}_i$ is a known sequence of time ordered second derivatives of $u(t)$ at times t_i, $i = 1,2,\ldots,n$. Further assume $\ddot{u}_i$ is constant on the interval $t_{i-1} < t \leq t_i$. Then $u_i(t)$ and $\dot{u}_i(t)$ on $t_{i-1} < t \leq t_i$ can be found as

$$u_i(t) = \frac{\ddot{u}_i \Delta t^2}{2} + \dot{u}_{i-1}\Delta t + u_{i-1} \qquad (23a)$$

$$\dot{u}_i(t) = \ddot{u}_i \Delta t + \dot{u}_{i-1} \qquad (23b)$$

where $\Delta t = t - t_{i-1}$

In applying the above equations, $u_i(t)$ and $\dot{u}_i(t)$ will represent the value of a control or state variable and its first time derivative. In case of $a(t)$ and $n_c(t)$, the first time derivatives need not be continuous, so they can be more simply prescribed by using equation 23b and equating $\dot{u}(t)$ to $a(t)$ or $n_c(t)$. An example of using the above method can be illustrated by prescribing a constant turn rate,

$\dot{\psi}$. Suppose ψ_0 and $\dot{\psi}$ are both zero at t_0. Then a constant turn rate, $\dot{\psi}$ could be achieved by t_1, and maintained until t_2 by specifying

$$\ddot{u}_1 = \frac{\dot{\psi}}{(t_1 - t_0)}, \quad \ddot{u}_2 = 0.$$

Since $u(t)$ and $\dot{u}(t)$ on $t_{i-1} < t \leq t_i$ are evaluated independently of t_i in equations 23a and 23b, the point in time when $\ddot{u}(t)$ switches from $\ddot{u}_i$ to $\ddot{u}_{i+1}$ does need to be explicitly defined with $\ddot{u}_i$. It will be useful to allow t_i to be optionally defined as the time when any state variable, control variable, their derivatives, or acceleration, say $v(t)$, crosses a threshold value, c, during the trajectory. Specifically, u_i and $\hat{u}_i$ are evaluated on the interval $t_{i-1} < t \leq t_i$; $t_i = \min(t)$ such that $v(t) \geq c$ (or $v(t) \leq c$). Thus, instead of a defining t_i explicitly, $v(t)$, c, and the desired logical operator can be defined. As an example of using this option, suppose an aircraft, initially in level flight at velocity V, is to increase engine speed to n_c', and accelerate to V_1. First initial conditions are defined for all variables except α and n_c which are solved at t_0 by selecting the known variables to be $\bar{u} = (V, \psi, \theta, \beta)$. The appropriate initial conditions and second derivatives prescribed for steady flight maintained for one second will be represented by: $\bar{u}_0 = (V_0,0,0,0)$; $\dot{\bar{u}}_0 = \ddot{\bar{u}}_1 = (0,0,0,0)$; $t_1 = 1$ sec. The values of the controls α and n_c, unknown prior to the solution, corresponding to the prescribed steady flight condition are found. After switching to the prescribed variable set $\bar{u} = (\psi, \theta, n_c, \beta)$, level acceleration will be accomplished by controlling n_c as follows:

$$\ddot{\bar{u}}_2 = (0,\ 0,\ \dot{n}_c,\ 0); \quad t_2 = \min(t) \text{ s.t. } n_c > n_c'$$

$$\ddot{\bar{u}}_3 = (0,\ 0,\ 0,\ 0); \quad t_3 = \min(t) \text{ s.t. } V(t) > V_1$$

$$\ddot{\bar{u}}_4 = (0,\ 0,\ -\dot{n}_c,\ 0); \quad t_4 = \min(t) \text{ s.t. } \dot{V}(t) < 0$$

$$\ddot{\bar{u}}_5 = (0,\ 0,\ 0,\ 0); \quad t_5 = 5 \text{ sec}$$

Recalling $n_c = \dot{u}(t)$, $\ddot{u}_2$ increases the engine speed, n_c, at a rate $\dot{n}_c$. After time t_2 the engine speed is constant at a value greater than n_c'. At time t_3, defined by $V(t) \geq V$, the engine speed decreases at a rate $-\dot{n}_c$ until positive acceleration ceases, $\dot{V}(t) \leq 0$. Then the aircraft maintains level flight for 5 seconds at a constant engine speed. The comparison of a variable with the threshold value to determine t_i will occur at time increments of Δt equal to the numerical integration stepsize. Therefore, the exact value of a variable at time t_i cannot generally be predicted prior to the trajectory solution.

RUNGE-KUTTA INTEGRATION

A fourth order Runge-Kutta algorithm will be used to numerically integrate the first order differential equations 14a through 16.

Dependent on the problem formulation, as many as seven simultaneous equations must be integrated. Let $\bar{y}$ represent the vector of state variables to be integrated, $\bar{x}$ the vector of prescribed state variables and control variables, and $\bar{f}(\bar{y}, \bar{x})$ the vector of ordinary differential equations in a given problem. The control variables in $\bar{x}$ may include those found as roots of equations 14a, 14b, and 14c so $\bar{x}$ will be, in general, a function of $\bar{y}$ as well as time. The general integration problem can be expressed as

$$\dot{\bar{y}} = \bar{f}(\bar{x}, \bar{y}), \qquad \bar{y}(t_0) = \bar{y}_0 \tag{24}$$

An approximation to $\bar{y}(t)$ at discrete points $t_i = t_0 + i\Delta t$, $i = 1,2,\ldots,n$, is desired where Δt is a constant step size. Let the approximation to the solution $\bar{y}(t_i)$ be noted $\bar{y}_i$. The following fourth order Runge-Kutta algorithm will be used to calculate $\bar{y}_{i+1}$.

$$\bar{y}_{i+1} = \bar{y}_i + \frac{\Delta t}{6}(\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4) \tag{25}$$

$$\bar{k}_1 = \bar{f}(\bar{y}_i, \bar{x}(t_i, \bar{y}_i))$$

$$\bar{k}_2 = \bar{f}(\bar{y}_i + \frac{1}{2}\Delta t\bar{k}_1, \bar{x}(t_i + \frac{1}{2}\Delta t, \bar{y}_i + \frac{1}{2}\Delta t\bar{k}_1))$$

$$\bar{k}_3 = \bar{f}(\bar{y}_i + \frac{1}{2}\Delta t\bar{k}_2, \bar{x}(t_i + \frac{1}{2}\Delta t, \bar{y}_i + \frac{1}{2}\Delta t\bar{k}_2))$$

$$\bar{k}_4 = \bar{f}(y_i + \Delta t\bar{k}_3, \bar{x}(t_i + \Delta t, \bar{y}_i + \Delta t\bar{k}_3))$$

Runge-Kutta algorithms are developed using Taylor series expansions of the unknown solution $y(t)$. Neglecting higher order terms in the expansion results in truncation error. An estimate of the error, e_t, given in [5] for fourth order integration is

$$e_t = \frac{16}{15}(y_{i+1,2} - y_{i+1,1}) \tag{26}$$

Here, $y_{i+1,1}$ is an approximation of the solution $y(t_{i+1})$ with truncation error e_t resulting from a step size Δt. The term $y_{i+1,2}$ is the approximation of $y(t_{i+1})$ based on two integration steps of size $\Delta t/2$. Truncation errors can be evaluated at intervals during integration and compared to a threshold error value for each state variable. If the error is exceeded, the size of Δt can be decreased. The differential equations, $\bar{f}(\bar{y}, \bar{x})$, must be evaluated four times for integration across Δt. This implies four evaluations of the forces and specific fuel consumption. Prescribed variables, only dependent on time, are to be evaluated at t_i, $t_i + \Delta t/2$, and t_{i+1}. The variables in $\bar{x}$ which are roots of equations must be found four times.

NEWTON-RAPHSON ALGORITHM

The Newton-Raphson algorithm will be used to find the control variables satisfying the set of equations 14b and 14c, when the velocity vector attitude is prescribed, or 14a, 14b, and 14c when total velocity vector is prescribed. In the first case, the controls

(β, α) or (ϕ, α), must be found; in the second case, (n_c, β, α) or (n_c, ϕ, α) are the controls to be found. Let f_1, f_2 and f_3 equal equations 14a, 14b, and 14c solved for zero and $\bar{x}$ be a vector containing the unknown control variables required to satisfy $\bar{f}(\bar{x}) = 0$. Given an initial estimate, $\bar{x}_0$, the approximation to the solution is iteratively incremented, $\bar{x}_{i+1} = \bar{x}_i + \Delta\bar{x}_i$ using

$$\Delta\bar{x}_i = - \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}^{-1} \begin{bmatrix} f_1(\bar{x}_i) \\ f_2(\bar{x}_i) \\ f_3(\bar{x}_i) \end{bmatrix} \tag{27}$$

where f_{jk} = the partial derivative of f_j with respect to control variable x_k.

When the velocity attitude is prescribed using $(\psi, \theta, n_c, \beta)$ or $(\psi, \theta, n_c, \phi)$, the problem reduces to solving two equations, f_2 and f_3, for the appropriate controls. Symmetric flight in the vertical plane and zero sideslip flight also reduce the number of equations to be solved. When the velocity vector is prescribed, f_2 is deleted from the problem for flight in the vertical plane while for zero sideslip flight f_2 is solved independently for ϕ using equation 19. Similarly, when velocity attitude is prescribed, only f_3 and α appear in equation 27 during symmetric flight in the vertical plane or zero sideslip flight.

Multiple roots may exist for any of the sets of equations to be solved. Using different initial estimates, $\bar{x}_0$, when starting the algorithm may provide the values of the multiple roots. Finding the multiple roots may be desired when starting the numerical solution at steady flight conditions, for example. The Newton-Raphson algorithm requires that the inverse of the partial derivative matrix exist. However, convergence is not guaranteed. If problems arise other root finding algorithms may be employed. Several alternative approaches are presented in [6]. The algorithm stops when the absolute value of elements in the increment vector, $\Delta\bar{x}_i$, are less than the elements in given vector $\bar{\varepsilon}$. Since controls have to be evaluated four times during integration across Δt, the choice of $\bar{\varepsilon}$ will be important in determining the time required to integrate the remaining differential equations.

In the cases where three equations are solved, nine of 12 possible partial derivatives (four possible control variables) appear in equation 27. The expressions for the partial derivatives are not presented here, but they include the force coefficient functions and partial derivative of these functions. Both the function values and partial derivatives have to be evaluated by interpolation at each iteration. Specifically, the partials to be evaluated are C_{L_α}, C_{D_α}, C_{C_β}, and $K_{T_{n_c}}$.

INTERPOLATION METHOD

Interpolation between discrete values of the force coefficient and specific fuel consumption functions arise in the solution of equations 14a through 16. Also, partial derivatives of these functions with respect to control variables will be required. Natural cubic spline interpolating functions will be employed to meet these requirements. To define these functions, assume n values of a function on one dimension, $f(x)$, are given at the base points x_i, $i = 1,2,\ldots,n$. Then $n-1$ third order cubic polynomials, $g_i(x)$, are found by requiring continuous first and second derivatives on (x_1,x_n). They are uniquely determined and called natural cubic splines when the second derivatives $g''(x_1)$ and $g''(x_n)$ are defined to be zero. The remaining $g_i'' = g''(x_i)$ are found from the solution of $n-2$ linear equations whose coefficients are determined by the function values and base points. The resulting coefficient matrix is tri-diagonal and easily solved by elimination and substitution. The problem formulation and solution methods are presented in [7].

Once the values for g_i'' are found, then $f(x)$ for $x_i < x < x_{i+1}$, is approximated as

$$
\begin{aligned}
f(x) \doteq\ & \frac{g''(x_i)}{6\Delta x_i}\left[(x_{i+1}-x)^3 - \Delta x_i^2(x_{i+1}-x)\right] \\
& + \frac{g''(x_{i+1})}{6\Delta x_i}\left[(x-x_i)^3 - \Delta x_i^2(x-x_i)\right] \qquad (28) \\
& + \frac{f(x_i)}{\Delta x_i}(x_{i+1}-x) + \frac{f(x_{i+1})}{\Delta x_i}(x-x_i) ,
\end{aligned}
$$

where $\Delta x_i = x_{i+1}-x_i$.

The derivative, $f'(x)$, follows from this equation.

To apply the cubic spline interpolation to functions defined on two dimensions, cubic splines will be calculated along both coordinates for every base point. A function $f(x_1,x_2)$ requires two second partial derivatives, $\partial^2 g_{ij}/\partial x_k^2$, $k = 1,2$, at each base point $(x_{1,i}, x_{2,j})$. These need only be calculated once and stored with the original function values for each base point. Equation 28 can be used to approximate an arbitrary point $f(x_1, x_2)$ where (x_1, x_2) is in the rectangle $x_{1,i} \leq x_1 \leq x_{1,j+1}$, $x_{2,j} \leq x_2 \leq x_{2,j+1}$, if a cubic spline parallel to one coordinate, say x_2, is defined. First the two points $f(x_1, x_{2,j})$ and $f(x_1, x_{2,j+1})$ are found with equation 28 applied twice along edges of the rectangle parallel to the x_1 coordinate using the stored function values and partial derivatives with respect to x_1. Next, two second partial derivatives in the x_2 direction $\partial^2 g(x_1, x_{2,j})/\partial x_2^2$ and $\partial g(x_1, x_{2,j+1})/\partial x_2^2$ are needed. These will be approximated by linear interpolation between partial derivatives in the x_2 direction known at the corner points of the rectangles.

$$\frac{\partial^2 g}{\partial x_2^2}(x_1, x_{2,k}) = \left[\frac{\partial^2 g_{i+1,k}}{\partial x_2^2}(x_1 - x_{1,i}) - \frac{\partial^2 g_{i,k}}{\partial x_2^2}(x_1 - x_{1,i+1})\right] \cdot \frac{1}{(x_{1,i+1} - x_{1,i})}, \quad k = j, j+1 \quad . \tag{29}$$

Once these second derivatives are calculated, then the two function values and the two derivatives required to evaluate $f(x_1, x_2)$ using equation 28 are known. The partial derivative of $f(x_1, x_2)$ with respect to x_2 can also be evaluated directly using $f'(x)$ found from equation 28.

APPROXIMATION OF AIRCRAFT ANGULAR VELOCITY

Calculating the aircraft angular velocity using equation 6 requires the time derivatives of the attitude control variables α, β, and ϕ. When any of thse are prescribed variables using the methods described earlier, the time derivatives are known. However, when α, β, and ϕ are found as roots of the force equations, then $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\phi}$ must be approximated. The time derivatives will be assumed linear and continuous so the angular velocity is also continuous. If x_i is the value of control variable at time t_i, then $\dot{x}_i$ is found by

$$\dot{x}_i = \frac{2(x_i - x_{i-1})}{(t_i - t_{i-1})} - \dot{x}_{i-1} \tag{30}$$

The initial value of the derivative, $\dot{x}_0$, needs to be defined to calculate the control variable derivatives of any time, t_i.

ATMOSPHERE MODEL

The atmospheric pressure, density, and speed of sound are needed during the solution to evaluate the forces, specific fuel consumption and Mach number. An arbitrary atmosphere can be constructed assuming a temperature versus altitude relation and a sea level pressure. Then pressure and density are found as functions of altitude by solving the hydrostatic equation and the ideal gas law simultaneously. The speed of sound, a, can be accurately modeled, see [8], as a function of absolute temperature, τ, using reference values a_0 and τ_0 by

$$a = a_0(\tau/\tau_0)^{1/2} \tag{31}$$

A standard atmosphere is frequently used in modeling aircraft flight. The NACA standard atmosphere as presented in [8] will be approximated by assuming a sea level temperature of 59°F decreasing at a rate of 3.56×10^{-3}°F/ft to -67.6°F at about 35.3 kft, where the temperature remains constant to well above conventional jet aircraft ceilings. At 59°F, the sea level sound velocity is 1,117 ft/sec, and the assumed pressure is 2116.2 lb/ft^2. With this information, the properties of air can be found and tabulated, then linear interpolation will be used for quickly evaluating the air properties for any altitude.

V. SUMMARY AND CONCLUSIONS

To assess aircraft tactics using quantitative models of weapons likely to engage an aircraft, a quantitative description of an aircraft's trajectory is required. In this report, a model conventionally used in aircraft trajectory analyses is formulated as a system of seven equations and eleven variables. The variables governed by the equations define the aircraft's position, velocity, fuel use, and attitude. A particular aircraft's capabilities are represented by the force and specific fuel consumption functions that appear in the equations. It is assumed these functions can be modeled by dimensionless coefficients defined on two dimensions.

To determine a unique trajectory governed by the equations, a set of variables must be prescribed as functions of time. Different sets of variables can be selected to conveniently prescribe different segments of a trajectory depending upon the aircraft flight condition or maneuver desired. Either the control variables, the attitude of the velocity vector, or the attitude and magnitude of the velocity vector can be prescribed with the latter two sets also including selected controls. The derivatives of the velocity attitude angles determine the flight path normal acceleration, a quantity useful for prescribing maneuvers. One of the velocity attitude angles can be replaced by normal acceleration in the set of prescribed variables if an additional equation governing the replaced variable is added to the model. To ensure aircraft acceleration and angular velocity are continuous, prescribed state and control variables should possess at least continuous first derivatives. Evaluating prescribed variables using a sequence of constant second time derivatives is one possible procedure that meets the continuity requirements.

Numerical solution of the equations involves application of algorithms for integrating first order differential equations, finding roots of nonlinear equations, and interpolation to approximate function values. During integration across a time interval, both the root-finding and interpolation algorithms must be applied at each derivative evaluation. Integration error can be estimated and the time interval reduced if desired. Each of the algorithms will be best implemented independently of the equations to be solved and in separate subroutines. The logic required to solve the system of equations will be reflected in the sequence of subroutine calls and the parameters that are passed at each call. This approach will reduce the effort necessary to solve the equations with different sets of aerodynamic force and engine data. Adding arbitrary functions for prescribing variables or extending the methods to solve other dynamics problems is simplified if the algorithms and equations of the trajectory generation model are well defined in the software implementation.

Aircraft trajectories found using the model in this report must be evaluated with respect to external constraints on the feasibility of the aircraft motion. These include limits on the aircraft structure, engine operation, and aircraft control. The operations manuals for a particular aircraft provide information on the flight limitations that must be observed. If maintaining aircraft control during a flight path is questioned, the implied moments can be calculated and compared to the maximum control moments available. The aircraft moment coefficient data and moment of inertia properties are needed for these calculations.

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